Research Statement—sarah-marie belcastro

I think of myself as a semi-generalist because I have knowledge and interests ranging across geometry, topology, algebra, and discrete mathematics. My graduate training was in algebraic geometry and I now concentrate on topological graph theory, but I also do research in other subfields of mathematics. What unifies my seemingly disparate work is a discrete or combinatorial perspective (e.g. using surface invariants to think about embedded graphs, computing knot polynomials recursively, or considering algebraic surfaces expressed in terms of convex polyhedra). This document describes my work and interests in mathematics (topological graph theory, algebraic geometry, and other subfields) and interdisciplinary fields (mathematics and fiber arts, feminist philosophy of science, mathematics and dance, and mathematical pedagogy). In the interest of saving space, not all of my publications are mentioned here or on my CV.

Part I: Mathematics

Topological Graph Theory

I study networks of vertices and edges (called *graphs*) and how to draw them on various surfaces (see Figure 1) without edges crossing (see Figure 2). A major open question in the field is how few holes a

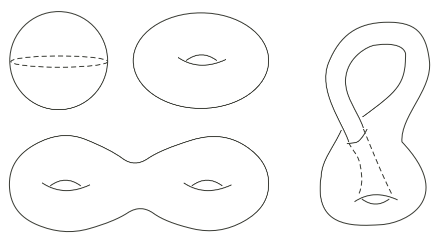

Figure 1: A sphere and torus above a 2-holed torus, adjacent to a Klein bottle.

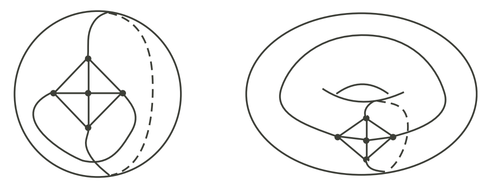

Figure 2: The graph K_5 must have edges crossing when drawn on the sphere (left), but can be drawn without edges crossing on the torus (right).

surface needs to have in order for one to draw a given graph there. I often investigate how few colors are necessary to color the edges of certain graphs such that no two edges of the same color touch. Interestingly, this can be influenced by the surfaces on which one can draw a graph without edges crossing!

A graph is composed of vertices and edges, where the combinatorial data thereby encoded is adjacency of vertices. Any graph can be drawn without edges crossing on some topological surface; such a drawing is called an *embedding* and introduces the additional structure of faces. It is desirable that all faces be open topological disks, and if so, the embedding is a *cellular* embedding. Topological graph theory is concerned with the same sorts of questions as ordinary graph theory, but focuses on the interplay between an embedding surface and other properties of the graph or class of graphs under study.

My interest in topological graph theory began in 2001, when I became aware of Grünbaum's Conjecture (1969). It says that every 3-regular graph polyhedrally embeddable on an orientable surface is three-edge colorable. (A graph is *polyhedrally* embeddable on a surface if it can be embedded with no face using an edge twice and no two faces sharing more than one edge.) The openness of the conjecture struck me, because while the Four Color Theorem implies that every 3-regular graph on the sphere is properly three-edge colorable, in thirty years no meaningful progress had been made for n-holed tori.

As I learned more, I became increasingly fascinated by examining graph coloring through topology and began to do research on edge-colorings and on embeddings of graphs on topological surfaces other than the sphere. Since it was posed, there have been significant contributions to the literature on Grünbaum's Conjecture. It is known by Vizing's Theorem (1964/5) that every 3-regular graph is either 3-edge colorable or 4-edge colorable, and a *snark* is a 4-edge colorable 3-regular graph of girth (smallest cycle length) at least 5 that is cyclically 4-edge connected. Only two snarks were known to embed on the torus until 2004, when Kaminski and I exhibited infinitely many snarks that embed on the torus and 2-holed torus with dual edge-width two [4]. That same year, Mohar and Vodopivec showed that for each $g \geq 3$ and $g = 1$, there exists a snark with a polyhedral embedding on the nonorientable surface $\mathbb{N}_g$. (The case $g = 2$ was addressed by Liu and Chen in 2012, and I add to this in [1].) In 2007, Kochol announced a genus-5 counterexample to Grünbaum's Conjecture (built from a snark given in [4]); at the same time, Albertson, Alpert, Haas, and I proved that toroidal triangulations of chromatic number other than 5 have Grünbaum colorings, thus affirming Grünbaum's Conjecture for most toroidal triangulations [3].

What follows is an indication of some of the many problems/questions in which I am interested.

• There are many open questions related to Grünbaum's Conjecture. Does the Conjecture hold for low-genus orientable surfaces? Does it hold for all graph embeddings with sufficiently large representativity?

• We could consider *generalized* Grünbaum colorings, in which non-facial triangles also use three edge colors. Which embedded triangulations with Grünbaum colorings do *not* have a generalized Grünbaum coloring?

• Gottlieb and Shelton showed that there exists a Grünbaum coloring of a planar triangulation with all color-induced subgraphs connected if and only if the triangulation is even. Does an analogous theorem hold for the torus? Is there a relationship to Hamilton cycles in the dual cubic graph?

• There are a number of other questions in which I am interested, including topics such as snark family genera, graph products and their effect on genus, using homology to get genus bounds, gaps in the genus range of homology-embedded signed graphs, and edge-coloring toroidal buckyballs.

Topological Graph Theory Publications

[1] Snarks on the Klein bottle, *Australasian Journal of Combinatorics*, 65(3) (2016), 232–250.

[2] Triangle-free Uniquely 3-Edge Colorable Cubic Graphs, with R. Haas, *Contributions in Discrete Mathematics*, 10(2) (2015), 39–44.

Using results from [5], we construct infinite families of triangle-free uniquely 3-edge colorable cubic graphs. Our nonplanar examples are the first found in at least 30 years; we also bound their genera.

[3] Grünbaum colorings of toroidal triangulations, with M. O. Albertson, H. Alpert, and R. Haas, *Journal of Graph Theory* 63(1) January 2010, 68–81.

[4] Families of dot-product snarks on orientable surfaces of low genus, with J. Kaminski, *Graphs and Combinatorics* 23(3) June 2007, 229–240.

We show that any snark that embeds on S_g with a noncontractible cycle of S_g that intersects the graph in exactly two edges will generate an infinite family of snarks on S_g.

[5] Counting edge-Kempe-equivalence classes for 3-edge-colored cubic graphs, with R. Haas, *Discrete Mathematics*, 325 (28 June 2014), 77–84.
and Edge-Kempe-equivalence graphs of class 1 k-regular graphs, with R. Haas, submitted.

For an edge-colored graph G, associate a graph $KE(G)$ with vertices corresponding to edge colorings of G and edges connecting two edge colorings of G that differ by an edge-Kempe switch. We give some results on the structure and number of components of $KE(G)$ when G is class 1 and k-regular.

[6] 1-factor covers of regular graphs, with M. Young, *Discrete Applied Mathematics*, 159(5) March 2011, 281–287.

This paper classifies those 3-regular graphs for which every covering by 1-factors reduces to a proper edge-coloring, and shows that there are finitely many $(k > 3)$-regular simple graphs with this property.

[7] Domino tilings of $2 \times n$ grids (or perfect matchings of grid graphs) on surfaces, preprint in revision.

We present an elegant and elementary approach to enumerating edge-labeled perfect matchings of $2 \times n$ grid graphs on topological surfaces.

[8] Minor-closed classes of signed graphs, with D. C. Slilaty, preprint.
and Topological minor-closed classes of signed graphs, with D. C. Slilaty, in preparation.

We define graphical minor-closed classes of signed graphs and find their forbidden minors. We then introduce embeddings where the graph signing must be compatible with the surface's homology, and investigate topological minor-closed classes of signed graphs and their forbidden minors.

[9] Hamilton circuits and Grünbaum colorings, with M. O. Albertson and R. Haas, in preparation.

We give necessary and sufficient conditions on color-induced subgraphs of Grünbaum colorings of planar triangulations such that the dual graphs have Hamilton circuits. R. Haas and I are attempting to obtain similar results for the torus.

[10] Parsimonious edge colorings on surfaces, in preparation.

I have weakly extended a result of M. O. Albertson and R. Haas on edge-coloring planar graphs to graphs on other surfaces, and am working to strengthen my extension.

Other Research Projects

Among my many interests are some areas of mathematics best explained by pictures. For example, convex geometry considers many-dimensional analogues of polygons such as in Figure 3. Knot theory studies idealized strings that have been knotted and then the ends fused; Figure 4 shows a knot formed using a particular pattern. The mathematics of paperfolding is concerned with the process indicated in Figure 5, where executing marked fold lines and angles produces an abstract object.

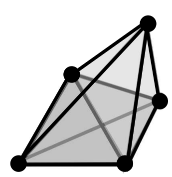

Figure 3: A four-dimensional simplex.

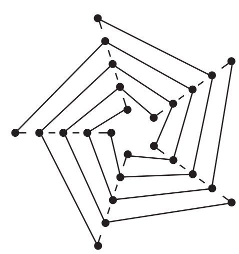

Figure 4: A (5,4) torus knot.

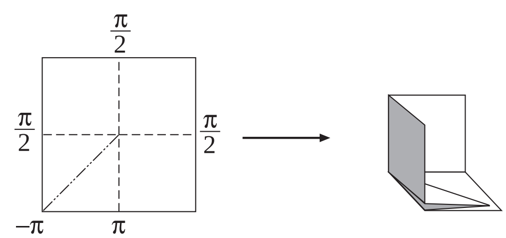

Figure 5: Folding a crease pattern into a three-dimensional corner.

Other Research Publications

[11] How to classify regular polytopes, with E. Peters, preprint.

We present a complete and self-contained classification of regular polytopes from the point of view of convex geometry. This includes some new geometric proofs.

[12] An elementary computation of the Conway polynomial for $(m, 3)$ and $(m, 4)$ torus links, with D. Rowland, to appear in *Journal of Combinatorial Mathematics and Combinatorial Computing.*

Using only the skein relation and some combinatorics, we find a closed form for the Conway polynomial of the $(m, 3)$ torus link and a trio of recurrence relations that define the Conway polynomial of any $(m, 4)$ torus link.

[13] Modelling the folding of paper into three dimensions using affine transformations, with T.C. Hull. *Linear Algebra and its Applications.* 348 (2002), 273–282.
and A mathematical model for non-flat origami, with T.C. Hull, in *Origami^3: Third International Meeting of Origami, Science, Mathematics and Education*, A K Peters, Ltd. (2002), 39–51.

We modelled 3D paperfolding using piecewise isometries and affine transformations of $\mathbb{R}^3$ and used the model to generalize a necessary condition for flat-foldability to 3D foldability. These two papers contain the same results but use different proof techniques. The LAA paper is widely cited and still relevant to physics and engineering researchers.

[14] Classifying frieze patterns without using groups, with T.C. Hull. *The College Mathematics Journal*, 33(2) March 2002, 93–98.

This paper gives a combinatorial classification of frieze patterns that does not use the group structure of isometries. (It becomes uninterestingly complicated when applied to wallpaper patterns.)

Algebraic Geometry

As a subfield of mathematics, algebraic geometry is rather broad, and can rely on techniques ranging from complex analysis to combinatorics. There are many descriptions of algebraic geometry, as well; one is as the study of geometric objects that can locally be described as zero-sets of polynomials. These objects are generally referred to as *varieties*.

My interests within algebraic geometry are algebraic surfaces and toric varieties. An *algebraic surface* is a two-complex-dimensional variety (topologists study them as 4-manifolds over the real numbers). A *toric variety* is a variety that contains the algebraic torus $(\mathbb{C}^*)^n$ as a dense subset. This definition gives no insight as to the really special property of toric varieties: they can be described combinatorially, in terms of polytopes or sets of polyhedral cones.

A *hypersurface* is the zero-set defined by a single equation. A *2-dim toric hypersurface* S is a nondegenerate hypersurface in a three-dimensional toric variety, defined by a Laurent polynomial $f = \sum c_i \vec{x_i}^{\vec{p_i}}$. The $\vec{p_i}$ correspond to points in $\mathbb{R}^3$; the convex hull of these points is a polytope Δ, called the *Newton polytope* associated to f. In particular, because the coefficients of f are not encoded in Δ, each Newton polytope corresponds to a family $\{\mathcal{S}\}$ of hypersurfaces. Using Δ, we obtain algebraic/geometric information about the parent variety $\mathbb{P}_\Delta$ and thereby $\{S\}$ as well.

In my dissertation, I developed a variety of techniques for analyzing elliptic fibrations on K3 2-dim toric hypersurfaces and used these techniques to compute an invariant (the Picard lattice $\mathrm{Pic}(S)$) for the generic member of each of the families of K3 2-dim toric hypersurfaces where $\mathbb{P}_\Delta$ is a weighted projective space (classified by M. Reid). A portion of this work appears in [15].

I have not actively worked in algebraic geometry for several years, but I still follow the literature and may return to some problems related to my dissertation at a future time.

Algebraic Geometry Publications

[15] Picard lattices of families of K3 surfaces, *Communications in Algebra.* 30(1), 61–82 (2002).

Part II: Interdisciplinary Work

Mathematics and Dance

In addition to taking dance classes for more than thirty years, I have pursued an academic interest in dance for the last twenty years and used mathematical ideas in my choreography. K. Schaffer and I gave a lecture/demonstration overviewing connections between mathematics and dance at the 2008 Joint Mathematics Meetings; this is summarized in [16]. More recently, I have studied R. Laban's *The Language of Movement*; in it, he invokes polyhedra, linkages, surfaces with boundary, and modular arithmetic. I have written 20 pages of notes and analysis on this topic that will become at least two papers.

Mathematics and Dance Publications

[16] Dancing Mathematics and the Mathematics of Dance, with K. Schaffer. *Math Horizons*, February 2011, pp. 16–20. Reprinted in *The Best Writing on Mathematics 2012*, Ed. Mircea Pitici, Princeton University Press, pp. 79–92.

Mathematical Knitting

There is a wealth of mathematical questions one can ask about knitting, with subjects ranging from analyzing the knitting process itself (two different mathematical models are shown in Figure 6) to studying different knitting techniques and finding accurate representations of mathematical objects. I am interested in and have worked with all of these aspects of knitting.

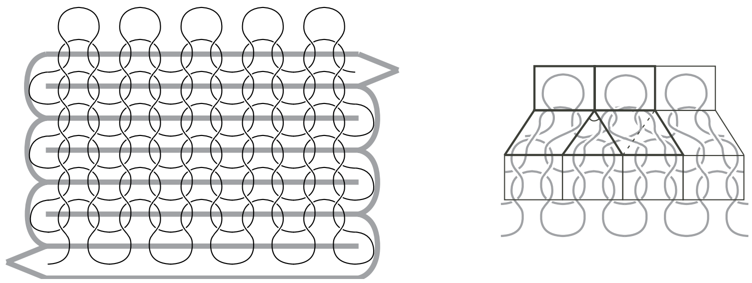

Figure 6: A topological model of knitting (left) and a geometric model of knitting (right).

I co-edited with C. Yackel the books *Making Mathematics with Needlework* (AK Peters, 2007), for which we wrote an introductory survey of mathematical research on fiber arts, and *Crafting by Concepts* (AK Peters, 2011). We have also written expository articles that illustrate advanced mathematics in knitting (not listed here).

Mathematical Knitting Research Publications

[17] Every Topological Surface Can Be Knit: A Proof, *Journal of Math. and the Arts*, June 2009, 67–83.

Using a rubber-sheet topology model of idealized knitting together with the classification of surfaces, I prove that any topological surface can be knit with a single strand of idealized yarn. I also give geometric considerations that inform algorithms for producing actual knitted topological surfaces.

[18] Only Two Knit Stitches Can Create a Torus, chapter in *Making Mathematics with Needlework*.

In this chapter, I classify all possible knit and purl stitches and determine all possible types of knitted fabric. It turns out that there are exactly two distinct knit stitches that produce conventional knitted fabric, independent of knitting flat or in the round.

[19] Stop Those Pants!, with C. Yackel, chapter in *Making Mathematics with Needlework*.

We give (with proof) a knitting construction for surfaces of uniform negative constant curvature, and focus on the hyperbolic pair of pants.

[20] Generalized Helix Striping, chapter in *Crafting by Concepts*.

I generalize helix striping from three to n colors and from stripe-heights of one to m, and analyze this construction from number-theoretic and braid-word perspectives.

[21] Knitting Torus Knots and Links, chapter in forthcoming (not yet titled) proceedings volume.

I describe an algorithmic method for preparing circular knitting needles so as to create infinite families of torus knots and torus links, and determine which torus knots and links can be achieved in this way.

Feminist Philosophy of Science

For the past twenty years, I have studied various feminist critiques of the scientific enterprise. The topics addressed by feminist theorists include representation issues, societal influence on the direction scientific research takes, and how the application of feminist theory can affect scientific knowledge. Within this landscape, my particular interest is in the feminist philosophy of science, specifically as applied to the pure physical sciences (mathematics included). Much work has been done in the past thirty or so years that clearly shows a range of effect that gender has had on the content and development of the social and biological sciences. Comparatively little has been written on the physical sciences and mathematics.

Feminist Philosophy of Science Publications

[22] Interpretations of Feminist Philosophy of Science by Feminist Physical Scientists, with J. M. Moran, NWSA Journal, 15 (1) Spring 2003, 20–33.

Our group of graduate-student physical scientists found that most critiques of the biological and social sciences do not apply to the physical sciences, and believes that feminist critiques of these sciences must be developed through dialogue between practicing physical scientists and feminist theorists.

[23] Intervening Discourse about Feminist Physical Sciences and Mathematics, preprint.

This is an outgrowth of work started in [22] and will be split into two or three separate papers. How can we reconcile or possibly mesh feminist theory with our traditional scientific training? This work explores the ways in which the pure physical sciences may be socially constructed, how the constitutive values of the pure physical sciences may be seen through the lens of gender, and how the way in which mathematics is communicated may affect what mathematics is done.

Mathematical Pedagogy

I am very interested in the teaching of mathematics. While working on my own pedagogy and discussing pedagogy with others, I have gained insights that others have agreed merit dissemination; this has frequently led to publication. In this section, no item descriptions are included because the titles are self-explanatory. Items [24], [25], [29], and [32] grew out of classroom experiences and [26] from an accessible research project. Items [30] and [31] originated in attempts to improve my teaching, and [27] and [28] came from attempts to answer questions raised in national conference sessions.

Mathematical Pedagogy Publications

[24] *Discrete Mathematics with Ducks*, textbook, AK Peters, 2012.

[25] Tablets versus IV—What's the Dose?, in preparation.

[26] Do the Twist! (on polygon-base boxes), with T. Veenstra, *The College Math. J.*, 47(5) November 2016, 340–345.

[27] Ask Questions to Encourage Questions Asked, *PRIMUS* 27(2), 2017, 171–178.

[28] To include more students, don't focus on contests—prepare for mathematics!, February 2004 *Mathematics Teacher*, Vol. 97 Issue 2, 84–86.

[29] The Devil is in the Culture: Why You Should Read The Number Devil and other Musings on Mathematical Education and Culture, with A. Howard, *Math Horizons* November 2002, 16–20 +29.

[30] Active Learning in Abstract Algebra: An Arsenal of Techniques, with L. Burton and M. McDermott, in *Innovations in Teaching Abstract Algebra*, MAA Notes 60, MAA, 2002, 3–9.

[31] A Teaching Discussion Group in Your Department—It Can Happen, with D. Shaw and D. Thiessen, *College Teaching* 50 (1),Winter 2002, 29–33.

[32] The Cantor Set Contains 1/4? Really?, with M. Green, *The College Math. J.*, 32 (1) 2001, 60–61.