Mathematics Triple Record - Academic Section

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1 Functions, Combinations, Hulls, and Dependence

The terms linear, affine, and convex can apply to all the terms in the title of this section.

1.1 Linear

Examples of linear functions are $f(x) = x$ and $f(x, y, z) = 2x + y - z$. There are two properties that they must fulfill: $f(ax) = af(x)$, and $f(x+y) = f(x)+f(y)$. It follows from the second property that $f\left(\sum x_i\right) = \sum f\left(x_i\right)$. Therefore, linear functions are of the form $f\left(x_1, ..., x_n\right) = \sum_{i=1}^{n} a_i x_i$. A linear combination of a set of x_is is $\sum_{i=1}^{n} a_i x_i$. A linear hull of a set of x_is is the set of all possible linear combinations of the x_is. The linear hull of 1 point is a line connecting the point and the origin, of 2 points is (in general) the plane connecting the points and the origin, etc. We have four different definitions for linear dependence:

1. Some x_i is a linear combination of the other x_js.
2. Some point can be written as a linear combination of all of the x_is in 2 different ways.
3. $\exists\, \lambda_i \mid \displaystyle\sum_{i=1}^{k} \lambda_i x_i = 0$ with $\lambda_i \neq 0$ for some i.
4. $\exists A, B \mid A \subseteq S, B \subseteq S, A \neq B, \mathit{lin}(A) = \mathit{lin}(B)$

We have proven that all of these definitions are equivalent.

1.2 Affine

Examples of affine functions are $f(x) = 2x + 3$ and $f(x, y, z) = 3x - 4y + \frac{z}{2} + 5$. They are linear functions with a constant term added. An affine combination of a set of x_is is $\sum_{i=1}^{n} \lambda_i x_i$ such that $\sum_{i=1}^{n} \lambda_i = 1$. An affine hull of a set of x_is is the set of all possible affine combinations of the x_is. The affine hull of 1 point is the point itself, of 2 points is a line connecting the points, of 3 points (in general) is the plane, etc. We have four different definitions for affine dependence:

1. Some x_i is an affine combination of the other x_js.
2. Some point can be written as an affine combination of all of the x_is in 2 different ways.

3. $\exists\, \lambda_i \mid \sum_{i=1}^{k} \lambda_i x_i = 0$ with $\sum_{i=1}^{k} \lambda_i = 0$ and $\lambda_i \neq 0$ for some i.

4. $\exists A, B \mid A \subseteq S, B \subseteq S, A \neq B, \mathit{aff}(A) = \mathit{aff}(B)$.

We have proven that all of these definitions are equivalent.

1.3 Convex

Convex combinations are affine combinations such that each $\lambda_i \in [0, 1]$. A set is convex if all segments with both endpoints in the set are completely contained in the set. The convex hull of a set of x_is is the set of all possible convex combinations of the x_is. The convex hull of two points is the segment connecting the points (without extending beyond), of three points (in general) is the triangle with those points as vertices, etc.

2 Boundary / Interior

2.0.1 Boundary

A boundary is a $(d-1)$-surface of an object in $\mathbb{R}^d$. Formally, a boundary is the intersection of the closure of a set and the closure of its complement.

2.0.2 Interior:

An interior is the the d-space in an object in $\mathbb{R}^d$. It is all points in a set that are not on the boundary of the set.

2.1 Relatives

2.1.1 Relative Boundary

The relative boundary of an ℓ-object in d-space, where $\ell \leq d$, is the absolute boundary of the object in ℓ-space.

2.1.2 Relative Interior

The relative interior of an ℓ-object in a d-space, where $\ell \leq d$, is the absolute interior of the object in ℓ-space.

2.2 Dot Product

A.K.A. *Standard Inner Product* or *Scalar Product*:

$$\langle\ ,\ \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

$$\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i$$

where $x = (x_1, x_2, \dots, x_d)$ and $y = (y_1, y_2, \dots, y_d)$.

3 Polytopes

A polytope is defined as the convex hull of a finite number of points. Polytopes are classified as d-polytopes, where d denotes the number of dimensions of the polytope, or the $\mathbb{R}^d$-space that the polytope exists in. All ℓ-dimensional substructures of a d-dimensional polytope should also be polytopes. Known polytopes are:

- d-cubes: These are formed by "dragging out" a $d-1$-cube, dragging the same distance through a new dimension every time.
- d-simplices: These are formed by "dragging out" a $d-1$-simplex the same distance through a new dimension, and converging to a point.
- d-prisms: These are formed by dragging out a $d-1$-dimensional polytope a distance I.
- d-cross polytopes: These are formed by finding the convex hull of d copies of [-1,1] arranged orthogonally and meeting at the origin.
- d-fold prisms: These are of the form $(P_{d-n} \times \underbrace{I \times \cdots \times I}_{n \text{ of these}})$
- d-fold pyramids: These are of the form $conv(P_{d-n}, \underbrace{*, \dots, *}_{n \text{ of these}})$, where the n points are affinely independent of P_{d-n} and of each other.

4 Other Figures

Flat: A flat is a space. The progression of flats is a point, line, plane, space, etc. We have two potential formal definitions. The first, Michael's, states that a k-flat is the linear hull of k linearly independent points in $\mathbb{R}^d$, with $k \leq d$. The other, Adam's states that a k-flat is the affine hull of $k+1$ affinely independent points in $\mathbb{R}^d$, with $k \leq d$.

Hyperplane: A hyperplane is a $(d-1)$-flat in $\mathbb{R}^d$, that is, a flat one dimension lower than the space in which it is contained. A supporting hyperplane is a hyperplane that intersects the relative boundary, but not the relative interior, of a convex set.

Halfspace: A halfspace is a hyperplane and all of the space to one side of it. It is defined as $K(a,c) = \{x \,|\langle a, x\rangle \leq c\}$. To get the other half of space, make a and c negative. When this is reevaluated, it comes out to $K(a,c) = \{x \,|\langle a, x\rangle \geq c\}$.