Geometric knitting

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Introduction

- I've been knitting for a few years, and I've found there's a lot of geometry hidden in it.
- The aim of this talk will be to look at some of the geometry involved in knitting. Hopefully this will also provide a concrete way to look at some quite abstract geometric concepts.

Introduction

- In particular, we'll look at how we can take a shape and build a knitting pattern for it.
- We'll do this in three ways:
  - first we'll approach the problem topologically, and construct patterns up to homeomorphism.
  - second we'll add a metric, then we'll place some restrictions on the space to make it easier to work with.
  - finally we'll turn to fully general metrics. Finding patterns here will be much harder, but I'll demonstrate a method which will let us do this algorithmically(if not actually easily).
- (No previous knowledge of knitting should be required.)

Topology - Knitting background

- The basic idea of knitting is to take a row of loops:

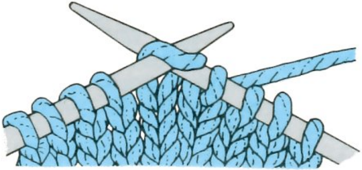


Fig 30

and to add a new row of loops passing through these:

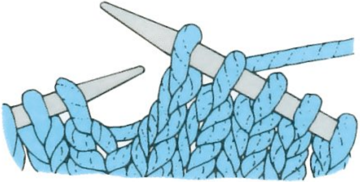


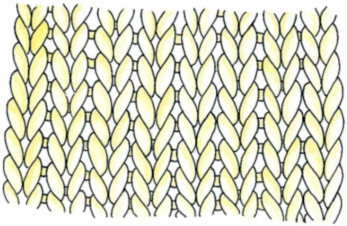
Fig 33

Topology - Knitting background

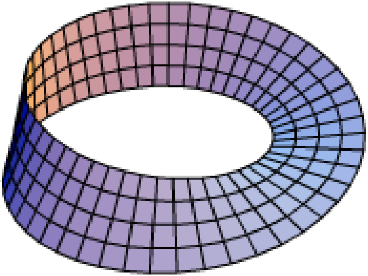
- Continuing in this way we build up a knitted fabric.
- If the new loops pass from front to back, we say the stitches are *knitted*, if they go from back to front, they are *purled*.
- A pattern of alternating knit and purl rows is called *stocking stitch*. This has all the loops pointing in the same direction, and is the basic knitted fabric.

Topology - Knitting background

- Because stitches are wider than they are tall, if we start with n stitches and knit m rows, we'll get roughly a $\frac{2}{3}n \times m$ rectangle.



Topology - Geometry background

- A surface is a space obtained by gluing discs together.
- These spaces are locally very simple, but can have interesting  global properties.

Topology - Main

- Given a topological surface, we can divide it up into a number of discs, glued along their boundaries.
- Then we can just knit as many rectangles as we have discs, and sew them together according to the gluing on their boundaries.

Topology - Main

- Mobius strip

Topology - Limitations

- Stocking stitch has a distinct front and back. For nonorientable surfaces, it'll be impossible to avoid joining the front of one disc to the back of another. There are other stitch patterns we can use in which the front and back look the same if we want to avoid this.
- This method only includes topological information, and it's likely to lead to a very uneven finished piece.

Homogeneous spaces

- We can introduce geometric information by putting a metric on our manifold. Initially we will require that the space has some symmetry to make it easier to deal with.

Homogeneous spaces - Knitting background

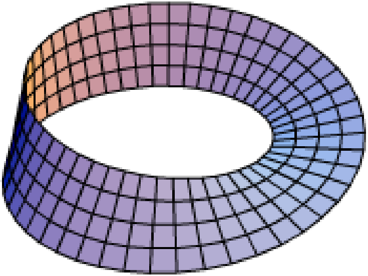
- One way to introduce shaping into knitting is by adding and removing stitches.
- There are several ways to do this, but probably the simplest way is to knit twice into one stitch, so we have two loops passing through one loop on the previous row.
- Similarly, we can decrease by knitting two stitches together, making one loop pass through two loops on the previous row.
- For technical reasons, we usually only increase and decrease on knit rows, leaving the purl rows plain.

Homogeneous spaces - Geometry background

- If G is a Lie group, a manifold M is a homogeneous space for G if G acts freely and smoothly on M. In that case, M locally has the form $D \times G$, where D is a disc of dimension $(\dim M - \dim G)$.
- Since we are interested in surfaces, G must have dimension 1, and hence can only be $\mathbb{R}$ or S^1.

Homogeneous spaces - Geometry background

- A good example of a G-bundle is the Mobius strip:



Homogeneous spaces - Geometry background(II)

- A Riemannian metric is an inner product on the tangent spaces of the manifold.
- This means that given a tangent vector on the surface we can measure it's length. Once we can say how 'fast' curves are moving, we can measure distances on the surface.
- Taking coordinates (x, y) on the surface gives us a basis on the tangent spaces, so we can write the metric as matrices. The we write

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = adx^2 + 2bdxdy + cdy^2 \tag{1}$$

Homogeneous spaces - Geometry background(II)

- A metric on a homogeneous space is G-invariant if moving tangent vectors around using the G-action does not change their lengths. In this case we can find coordinates such that the metric has the form

$$g = dx^2 + f(x)^2 dy^2 \qquad 0 \leq y \leq 1,\ 0 \leq x \leq 2N \tag{2}$$

- Here the y coordinate is parallel to the orbits of G, so this says all the fibres $\{x = b\}$ are homogeneous in y. However their lengths may vary with f.
- The x coordinate is orthogonal to the orbits of G, and this tells us the contours $\{y = c\}$ all have the same length.

Homogeneous spaces - Main

- We can approach knitting such surfaces as follows: First, we divide the surface up into charts parametrized by I^2 with coordinates in the desired form. Then we'll identify the knitted rows with the fibres $\{x = b\}$, and the 'columns' with the contours $\{y = c\}$. We can use increases and decreases to vary the lengths of the fibres as b varies.

Homogeneous spaces - Main

- The fibres will have length

$$
\begin{aligned}
l(x) &= \int_0^1 \sqrt{g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)}\, dy &\quad (3)\\
&= \int_0^1 f(x)\, dy &\quad (4)\\
&= f(x) &\quad (5)
\end{aligned}
$$

- In particular the first row should have $[\frac{3}{4}f(0)]$ stitches, and on the $(2n)$th (knit) row, we will need $[\frac{3}{4}(f(n+2) - f(n))]$ increases(in this value is negative, we interpret this as a number of decreases)
- The increases/decreases should be spaced as evenly as possibly, to reflect the symmetry of the fibres.

Homogeneous spaces - Main

- This gives us enough information to write down a pattern:
  - Cast on $[\frac{3}{4}f(0)]$ stitches.
  - $$(\mathrm{K}[\frac{3}{4}\frac{f(n)}{f(n+2)-f(n)}]-1, \text{ inc } 1)\times[\frac{3}{4}(f(n+2)-f(n)], \text{ K to end} \tag{6}$$
    $$\text{if } f(n+2)-f(n)\geq 0 \tag{7}$$
    $$(\mathrm{K}[\frac{3}{4}\frac{f(n)}{f(n)-f(n+2)}]-2, \text{ inc } 1)\times[\frac{3}{4}(f(n)-f(n+2)], \text{ K to end} \tag{8}$$
    $$\text{if } f(n+2)-f(n)\leq 0 \tag{9}$$
  - Purl next row
  - Repeat these two rows until $n=2N$, then bind off.

Homogeneous spaces - Main

- Example - Torus

Homogeneous spaces - Limitations

- This encodes all the geometry of the surface, but the symmetry requirement restricts the surfaces we can apply this method to.
- We're limited by the number of increases and decreases we can work in a row - we can at most double or halve the number of stitches, so we need to make sure our f doesn't vary too rapidly.

Homogeneous spaces - Limitations

- It's possible to generalise this method to deal with metrics

$$g = dx^2 + f(x,y)^2 dy^2 \tag{10}$$

  Unfortunately, this might require large areas of overlap and a large number of charts. This is OK for geometry, but is messy for knitting.
  Instead we'll loosen our restriction to allow us more freedom in choosing coordinates.

General metrics - Knitting background

- We'll use one more shaping technique - short rows.
- Starting on a purl row, purl part way across the row, turn, and knit part way back. Turn again, and purl to the end of the row.
- The effect of this is to add an extra row across part of the knitted piece.

- In practice, we do something more complicated at the ends of the short row to make them tidier, but this won't affect the geometry.

General metrics - Main

- Now suppose we have a more general metric,

$$g = a(x,y)^2 dx^2 + b(x,y)^2 dy^2 \qquad 0 \leq y \leq 1,\ 0 \leq x \leq 2N \tag{11}$$

- Since there is no $dxdy$ term, the coordinates are still orthogonal, but now the length of the columns can vary too, and neither coordinate is symmetric.
- (Disclaimer) I'll indicate a method for approaching this situation. I'll avoid the technical details, but try to emphasise the general approach involved.

General metrics - Main

- We start by casting on $n = [\frac{3}{4}\int_0^1 b(0,y)\,dy]$ stitches, representing the points $(x,y) = (0, \frac{k}{n})$, $0 \leq k \leq n$.
- Next we calculate

$$[\frac{3}{4}\int_0^{y_i} b(x,y)\,dy] - [\frac{3}{4}\int_0^{y_{i-1}} b(x,y)\,dy] - 1 \qquad (12)$$

  for each point.
- If this is $+1$, we work an increase into the relevant stitch, if it's -1, we work a decrease.
- If we increase or decrease, we need to add or remove an entry into out list of points to represent the stitch we create or destroy. Where we add a stitch, the point we add should be the midpoint of the two adjacent points in the list.

General metrics - Main

- Then, for each point we calculate $[\int_0^2 a(x, y)\, dx)] - 1$, which tells us how far we move along the contour as x moves from 0 to 2.
- Where this is $+1$, we work a short row over the correspoinding stitches, if it is -1 over some of the stitches, we work a short row over the others.
- We will need to keep track of $\int_0^2 a(x, y)\, dx - 1$ for each point-this tells us how far this stitch is out of position horizontally. We will add a new contribution to this on each row, and where it becomes too large or small, we will use short rows to correct the discrepancy.
- Continue in this way - at each step, we integrate b to decide where we need to increase and decrease, and integrate a to find how much we have moved out of position in the x direction, and if we are too far out of position we correct this using short rows.

General metrics - Limitations

- Again, we are limited by how rapidly we can increase and decrease. However, it is possible to work several short rows, on top of each other, so a simple modification to the above procedure will take care of all but the most extremely rapidly varying functions b.
- This process could be carried out entirely automatically. However, before we begin, we do need to be able to write down the metric on our surface in the first place.
- However, from observing how the algorithm works, we could at least pick out roughly what shaping is needed to give a particular shape. With a little judgement it would be possible to work out a rough pattern without doing the calculations explicitly.

Conclusion

- We've seen how we can apply geometric techniques to shape knitted surfaces. There are many other kinds of maths which come up in other aspects of knitting - being centred around patterns, knitting is intrinsically mathematical.
- Knitting is a nice concrete way to deal with geometrical concepts - we've seen manifolds, orientability, Riemannian metrics and homogeneous spaces, and all in a concrete way which is accessible to non-mathematicians.
- It's also a relaxing thing to do during tea-breaks.

Conclusion - Pretty pictures

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Conclusion - Pretty pictures

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Conclusion - Thanks!

- Thanks to:
  - learn2knit.co.uk
  - howstuffworks.com
  - ToroidalSnark.net
  - Knitty.com
  - Dr.Hinke Osinga ... all of whom I've stolen graphics from.
  - and Tim, for making Latex work.
- And thanks for listening!